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The Determination of Satellite Clock Corrections for Precise Point Positioning with CDMA GNSS Signals

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Abstract. This paper deals with the algebraic principles of Precise Point Positioning with CDMA GNSS signals. It covers specifically the case of a network solution that is the determination of precise satellite corrections by joint processing of measurements generated by a network of ground stations. These satellite corrections are further delivered to user receivers via communication channels and are used to compute user coordinates with errors, usually not exceeding 1 cm. The determination of these corrections requires high precision and is carried out by processing phase measurements with resolved integer ambiguities. Resolution of phase ambiguities considerably improves the accuracy of estimated corrections and, at the same time, sharply reduces the time required to achieve centimeter-level positioning accuracy in the user solution. Algebraic principles of the user solution with integer ambiguity resolution of phase measurements with GNSS CDMA signals were studied in the previously published paper of the authors [1].

Keywords: precise point positioning (PPP), ambiguity resolution (AR), Float PPP, Integer PPP

Introduction

For the methodology and notation as well as key concepts and terminology used in this paper, the readers are referred to the previous publication of the authors [1]. In fact, in both cases of user and network solutions, the main challenge of measurement processing is in dealing with the rank deficiency of linear systems obtained by linearizing nonlinear mathematical models of code and phase measurements preserving integer nature of phase ambiguities.

The integer nature of phase ambiguities allows one to apply ambiguity resolution. This entails considerable increase in the accuracy of estimated corrections and, therefore, dramatic reduction in time required to achieve centimeter-level accuracy of user's positional solution.

Mathematical models of code and phase measurements in Integer Precise Point Positioning (PPP) at ionosphere-free GNSS frequencies with CDMA

Mathematical models of the code $\rho^j_{ifr.i}$ and the phase $L^j_{ifr.i}$ measurements and of the Melbourne–Wübbena combination mw^j_i in Integer PPP algorithms at ionosphere-free GNSS frequencies result in a similar efficiency of ambiguity resolution as with the models based on initial frequencies. Such models were already introduced in the previous paper of the authors [1] and read as follows:

$$\begin{aligned}
\rho^j_{ifr.i} &= R^j_i + w^j_i \Delta D_i + dT_{\rho.ifr.i} - dt^j_{\rho.ifr.i} + \xi^j_{\rho.ifr.i} \\
L^j_{ifr.i} &= R^j_i + w^j_i \Delta D_i + dT_{L.ifr.i} - dt^j_{L.ifr.i} - \\
&- \lambda_{\Delta nifr} N1^j - \lambda_{n_2 ifr} N^j_{mw} + \xi^j_{L.ifr.i} \\
mw^j_i &= b_{mw} - b^j_{mw} - \lambda_{mw} N^j_{mw} + \xi^j_{mw.i} \\
j &= \overline{1, J_i}
\end{aligned} \tag{1}$$

The notations are identical to [1].

The network solution

In the description of a network solution the following notations are used: J_i is the number of space vehicles (SV) in CDMA GNSS which are simultaneously visible by the entire network of ground stations at the i-th epoch; M is the number of ground network stations; $J_{m.i}$ is the number of SVs visible by the m-th ground station $m = \overline{1, M}$ at the i-th epoch. Table 1 is taken from [2] and is somewhat reduced for brevity. This table gives an example of satellite visibility by a network of GPS ground stations; further, it is referred to as a 'scenario matrix'.

In Table 1 "1" corresponds to visible SVs, while "0" – to invisible SVs. The meaning of indices beside the units and shadowing of several units will become clear further. For the scenario given in Table 1, at the i-th epoch M=7, J_i=12, $J_{m.i}$ equals the number of units in the scenario matrix lines, i.e., to the number of SVs visible by each m-th ground station at the i-th epoch. $J_{\Sigma.k} = \sum_{m=1}^{M} J_{m.k}$ (note that for the scenario of Table 1, $J_{\Sigma.i}$ = 32).

The purpose of a network solution is to estimate the ionosphere-free code clock offsets $dt^j_{\rho.ifr.i}$, phase clock offsets $dt^j_{L.ifr.i}$ and hardware biases b^j_{mw} of the Melbourne–Wübbena combination ($j = \overline{1, J_{m.i}}$) for the whole set J_i of SVs visible by all ground stations in the network using the measurements of all the M ground stations comprising the network. In this case, the coordinates of stations and J_i observed satellites are assumed to be known with high accuracy. Subject to this, the system of nonlinear equations (1) of ionosphere-free measurements of the m-th station of the network ($m = \overline{1, M}$) can be represented in the network solution in the following linearized form (instead of the lower ifr index, the index m indicating the station number is applied):

$$\begin{aligned}
\Delta\rho^j_{m.i} &= w^j_{m.i} \Delta D_{m.i} + dT_{\rho.m.i} - dt^j_{\rho.i} + \xi^j_{\rho.m.i} \\
\Delta L^j_{m.i} &= w^j_{m.i} \Delta D_{m.i} + dT_{L.m.i} - dt^j_{L.i} - \\
&- \lambda_{\Delta nifr} N1^j_m - \lambda_{n_2 ifr} N^j_{mw.m} + \xi^j_{L.m.i} \\
\Delta mw^j_{m.i} &= b_{mw.m} - b^j_{mw} - \lambda_{mw} N^j_{mw.m} + \xi^j_{mw.m.i} \\
m &= \overline{1, M}, \ j = \overline{1, J_{m.i}}
\end{aligned} \tag{2}$$

where $\Delta\rho^j_{m.i} = \rho^j_{m.i} - R^j_{m.i}$, $\Delta L^j_{m.i} = L^j_{m.i} - R^j_{m.i}$, $\Delta mw^j_{m.i} = mw^j_{m.i}$ are the residuals of ionosphere-free combinations of the code $\rho^j_{m.i}$ and the phase $L^j_{m.i}$ as well as the Melbourne-Wübbena combinations $mw^j_{m.i}$ for measurements of the m-th station; $R^j_{m.i}$ is the geometric distance between the m-th station $m = \overline{1, M}$ and j-th SV $j = \overline{1, J_{m.i}}$.

Table 1. SV visibility by seven network stations: 1 – SV is visible, 0 – SV is invisible

	m	PRN numbers of GPS SVs visible by the entire network of ground stations												$J_{m.i}$
		1	2	3	10	16	17	21	22	23	26	27	31	
		Numbers of SVs in their positions in the united array												
		1	2	3	4	5	6	7	8	9	10	11	12	
Ground network stations	1	0	0	$1_{1,1}$	0	0	1_{2}	1_{3}	0	1_{4}	$1_{5,2}$	0	0	5
	2	$1_{6,3}$	0	1_{7}	0	0	$1_{8,4}$	$1_{9,5}$	0	1_{10}	0	0	0	5
	3	1_{11}	0	1_{12}	0	0	1_{13}	$1_{14,6}$	0	$1_{15,7}$	0	0	0	5
	4	0	0	0	0	$1_{16,8}$	0	0	$1_{17,9}$	0	0	0	$1_{18,10}$	3
	5	0	$1_{19,11}$	0	$1_{20,12}$	0	0	0	0	0	$1_{21,13}$	1_{22}	0	4
	6	0	0	1_{23}	0	0	1_{24}	$1_{25,14}$	0	1_{26}	$1_{27,15}$	$1_{28,16}$	$1_{29,17}$	7
	7	0	1_{30}	0	1_{31}	$1_{32,18}$	0	0	0	0	0	0	0	3

The system of linearized equations (2) for the network solution can be rewritten with the use of the matrix notations:

$$\underset{3J_{\Sigma.i}\times 1}{\boldsymbol{Y}_{net.i}} = \underset{3J_{\Sigma.i}\times nx}{\boldsymbol{H}_{net.i}} \cdot \underset{nx_i\times 1}{\boldsymbol{x}_{net.i}} + \underset{3J_{\Sigma.i}\times 1}{\boldsymbol{\Xi}_{net.i}} \tag{3}$$

where $\underset{3J_{\Sigma.i}\times 1}{\boldsymbol{Y}_{net.i}} = \left[\underset{1\times J_{\Sigma.i}}{(\boldsymbol{\Delta\rho}_i)^T} \quad \underset{1\times J_{\Sigma.i}}{(\boldsymbol{\Delta L}_i)^T} \quad \underset{1\times J_{\Sigma.i}}{(\boldsymbol{\Delta mw}_i)^T} \right]^T$ is the vector of observations, in which $\underset{J_{\Sigma.i}\times 1}{\boldsymbol{\Delta\rho}_i} = \left[\underset{1\times J_{1.i}}{\boldsymbol{\Delta\rho}_{1.i}^T} \quad \underset{1\times J_{2.i}}{\boldsymbol{\Delta\rho}_{2.i}^T} \quad \cdots \quad \underset{1\times J_{M.i}}{\boldsymbol{\Delta\rho}_{M.i}^T} \right]^T$ is the vector of ionosphere-free code residuals arranged in the order of stations and in order of SVs for each station;

$$\underset{J_{m.i}\times 1}{\boldsymbol{\Delta\rho}_{m.i}} = \left[\Delta\rho_{m.i}^1 \quad \Delta\rho_{m.i}^2 \quad \cdots \quad \Delta\rho_{m.i}^{J_{m.i}} \right]^T, \qquad m = \overline{1,\, M}$$

is the vector of ionosphere-free code residuals for the m-th station ordered by SVs; $\Delta\rho_{m.i}^j$, $m = \overline{1,\, M}$, $j = \overline{1,\, J_{m.i}}$ is the residual of ionosphere-free code measurement of the j-th SV at the m-th station; $\underset{J_{\Sigma.i}\times 1}{\boldsymbol{\Delta L}_i} = \left[\underset{1\times J_{1.i}}{\boldsymbol{\Delta L}_{1.i}^T} \quad \underset{1\times J_{2.i}}{\boldsymbol{\Delta L}_{2.i}^T} \quad \cdots \quad \underset{1\times J_{M.i}}{\boldsymbol{\Delta L}_{M.i}^T} \right]^T$ is the vector of ionosphere-free carrier phase residuals arranged in the order of stations and in the order of SVs for each station; $\underset{J_{m.i}\times 1}{\boldsymbol{\Delta L}_{m.i}} = \left[\Delta L_{m.i}^1 \quad \Delta L_{m.i}^2 \quad \cdots \quad \Delta L_{m.i}^{J_{m.i}} \right]^T$, $m = \overline{1,\, M}$ is the vector of ionosphere-free carrier phase residuals of the m-th station arranged in the order of SVs; $\Delta L_{m.i}^j$, $m = \overline{1,\, M}$, $j = \overline{1,\, J_{m.i}}$ is the residual of ionosphere-free carrier phase measurement of the j-th SV at the m-th station; $\underset{J_{\Sigma.i}\times 1}{\boldsymbol{\Delta mw}_i} = \left[\underset{1\times J_{1.i}}{\boldsymbol{\Delta mw}_{1.i}^T} \quad \underset{2\times J_{1.i}}{\boldsymbol{\Delta mw}_{2.i}^T} \quad \cdots \quad \underset{1\times J_{M.i}}{\boldsymbol{\Delta mw}_{M.i}^T} \right]^T$ is the vector of residuals of Melbourne-Wübbena combinations arranged in the order of the stations and in the order of SVs for each station. $\underset{J_{m.i}\times 1}{\boldsymbol{\Delta mw}_{m.i}} = \left[\Delta\, mw_{m.i}^1 \quad \Delta\, mw_{m.i}^2 \quad \cdots \quad \Delta\, mw_{m.i}^{J_{m.i}} \right]^T$, $m = \overline{1,\, M}$ is the vector of residuals of Melbourne-Wübbena combinations at the m-th station arranged in the order of SVs; $\Delta\, mw_{m.i}^j$, $m = \overline{1,\, M}$, $j = \overline{1,\, J_{m.i}}$ is the residual of the Melbourne-Wübbena combination of the j-th SV at the m-th station.

$$\underset{nx_i\times 1}{\boldsymbol{x}_{net.i}} = \Big[\underset{1\times M}{\boldsymbol{\Delta D}_i^T} \quad \underset{1\times M}{\boldsymbol{dT}_{\rho.i}^T} \quad \underset{1\times M}{\boldsymbol{dT}_{L.i}^T} \quad \underset{1\times M}{\boldsymbol{B}_{mw.i}^T} \quad \underset{1\times J_i}{\boldsymbol{dt}_{\rho.i}^T} \quad \underset{1\times J_i}{\boldsymbol{dt}_{L.i}^T} \quad \underset{1\times J_i}{\boldsymbol{b}_{mw.i}^T} \quad \underset{1\times J_{\Sigma.i}}{\boldsymbol{N1}_i^T} \quad \underset{1\times J_{\Sigma.i}}{\boldsymbol{N}_{mw.i}^T} \Big]^T \tag{4}$$

is the vector of the estimated variables in the network solution with the dimension nx_i=$4M$+$3J_i$+$2J_{\Sigma.i}$, where $\underset{1\times M}{\boldsymbol{\Delta D}_i^T} = \left[\Delta D_{1.i}^T \quad \Delta D_{2.i}^T \quad \cdots \quad \Delta D_{M.i}^T \right]^T$ is the M-vector of the uncompensated wet component of zenith tropospheric delays (m) at the locations of ground stations; $\underset{1\times M}{\boldsymbol{dT}_{\rho.i}^T}$ is the M-vector of receiver ionosphere-free code clock offsets for all the M stations; $\underset{1\times M}{\boldsymbol{dT}_{L.i}^T}$ is the M-vector of receiver ionosphere-free phase clock offsets for all the M stations of the ground network; $\underset{1\times M}{\boldsymbol{B}_{mw.i}^T}$ is the M-vector of receiver hardware biases of the Melbourne–Wübbena combinations for the M stations of the ground network; $\underset{1\times J_i}{\mathbf{dt}_{\rho.i}^T}$ is the J_i-vector of ionosphere-free code clock offsets for J_i SVs visible by the entire network of ground stations at the i-th epoch; $\underset{1\times J_i}{\mathbf{dt}^T_{L.i}}$ is the J_i-vector of ionosphere-free code clock offsets for J_i SVs visible by the entire network of ground stations at the i-th epoch; $\underset{1\times J_i}{\boldsymbol{b}^T_{mw.i}}$ is the J_i-vector of satellite hardware biases of Melbourne-Wübbena combinations for J_i SVs visible by the entire network of ground stations at the i-th epoch; $\underset{1\times J_{\Sigma.i}}{\boldsymbol{N}1^T_i}$ is the $J_{\Sigma.i}$-vector of integer ambiguities $N1$ included in the system (2) arranged in the order of stations and in the order of SVs for each station at the i-th epoch; $\underset{1\times J_{\Sigma.i}}{\boldsymbol{N}^T_{mw.i}}$ is the $J_{\Sigma.i}$-vector of integer ambiguities N_{mw} included in the system (2) arranged in the order of stations and in the order of SVs for each station at the i-th epoch; $\underset{3J_{\Sigma.i}\times 1}{\boldsymbol{\Xi}_{net.i}} = \begin{bmatrix} \underset{1\times J_{\Sigma.i}}{\boldsymbol{\Xi\rho}^T_i} & \underset{1\times J_{\Sigma.i}}{\boldsymbol{\Xi L}^T_i} & \underset{1\times J_{\Sigma.i}}{\boldsymbol{\Xi mw}^T_i} \end{bmatrix}^T$ is the vector of measurement errors of the ionosphere-free code $\underset{1\times J_{\Sigma.i}}{\boldsymbol{\Xi\rho}^T_i}$, ionosphere-free phase $\underset{1\times J_{\Sigma.i}}{\boldsymbol{\Xi L}^T_i}$ and of Melbourne-Wübbena $\underset{1\times J_{\Sigma.i}}{\boldsymbol{\Xi mw}^T_i}$ combination, which are formed according to the same principle as the vector $\underset{3J_{\Sigma.i}\times 1}{\boldsymbol{Y}_{net.i}}$.

$$\underset{3J_{\Sigma.i}\times nx_i}{\boldsymbol{H}_{net.i}} = \begin{bmatrix} \underset{3J_{\Sigma.i}\times M}{\boldsymbol{HW}_i} & \underset{3J_{\Sigma.i}\times 3(M+J_i)}{\boldsymbol{d}_i} & \underset{3J_{\Sigma.i}\times 2J_{\Sigma.i}}{\boldsymbol{\Lambda}_i} \end{bmatrix} \qquad (5)$$

is the design matrix for the observation vector $\underset{3J_{\Sigma.i}\times 1}{\boldsymbol{Y}_{net.i}}$ and the vector of the estimated variables $\underset{nx_i\times 1}{\boldsymbol{x}_{net.i}}$ (4), where $\underset{3J_{\Sigma.i}\times M}{\boldsymbol{HW}_i} = \begin{bmatrix} \underset{M\times J_{\Sigma.i}}{\boldsymbol{Hw}^T_i} & \underset{M\times J_{\Sigma.i}}{\boldsymbol{Hw}^T_i} & \underset{M\times J_{\Sigma.i}}{\mathbf{0}^T_i} \end{bmatrix}^T$ is the $(3J_{\Sigma.i}\times M)$-matrix of mapping functions $w^j_{m.i}$ ($m=\overline{1,M}$, $j=\overline{1,J_{m.i}}$), which are used to convert zenith tropospheric delays (m) into slant delays for actual elevation angles of satellites visible at the m-th station at the i-th epoch;

$$\underset{J_{\Sigma.i}\times M}{\boldsymbol{Hw}_i} = \begin{bmatrix} w^1_{1.i} & \cdots & w^{J_{1.i}}_{1.i} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & w^1_{M.i} & \cdots & w^{J_{M.i}}_{M.i} \end{bmatrix}^T;$$

$$\underset{3J_{\Sigma.i}\times 3(M+J_i)}{\boldsymbol{d}_k} = \begin{bmatrix} \underset{J_{\Sigma.i}\times M}{\mathbf{1}} & \underset{J_{\Sigma.i}\times M}{\mathbf{0}} & \underset{J_{\Sigma.i}\times M}{\mathbf{0}} & \underset{J_{\Sigma.i}\times J_i}{-\boldsymbol{U}} & \underset{J_{\Sigma.i}\times J_i}{\mathbf{0}} & \underset{J_{\Sigma.i}\times J_i}{\mathbf{0}} \\ \underset{J_{\Sigma.i}\times M}{\mathbf{0}} & \underset{J_{\Sigma.i}\times M}{\mathbf{1}} & \underset{J_{\Sigma.i}\times M}{\mathbf{0}} & \underset{J_{\Sigma.i}\times J_k}{\mathbf{0}} & \underset{J_{\Sigma.i}\times J_i}{-\boldsymbol{U}} & \underset{J_{\Sigma.i}\times J_i}{\mathbf{0}} \\ \underset{J_{\Sigma.i}\times M}{\mathbf{0}} & \underset{J_{\Sigma.i}\times M}{\mathbf{0}} & \underset{J_{\Sigma.i}\times M}{\mathbf{1}} & \underset{J_{\Sigma.i}\times J_i}{\mathbf{0}} & \underset{J_{\Sigma.i}\times J_i}{\mathbf{0}} & \underset{J_{\Sigma.i}\times J_i}{-\boldsymbol{U}} \end{bmatrix};$$

$$\underset{J_{\Sigma.i}\times M}{\mathbf{1}} = \begin{bmatrix} \underset{J_{1.i}\times 1}{\mathbf{1}} & \underset{J_{1.i}\times 1}{\mathbf{0}} & \cdots & \underset{J_{1.i}\times 1}{\mathbf{0}} \\ \underset{J_{2.i}\times 1}{\mathbf{0}} & \underset{J_{2.i}\times 1}{\mathbf{1}} & \cdots & \underset{J_{2.i}\times 1}{\mathbf{0}} \\ \vdots & \vdots & \ddots & \vdots \\ \underset{J_{M.i}\times 1.i}{\mathbf{0}} & \cdots & \cdots & \underset{J_{M.i}\times 1}{\mathbf{1}} \end{bmatrix};$$

$$\underset{J_{m.i}\times 1}{\mathbf{1}} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},\ m=\overline{1,M};$$

$\underset{J_{\Sigma.i}\times J_i}{\boldsymbol{U}_i} = \begin{bmatrix} \underset{J_i\times J_{1.i}}{\boldsymbol{U}^T_{1.i}} & \underset{J_i\times J_{2.i}}{\boldsymbol{U}^T_{2.i}} & \cdots & \underset{J_i\times J_{M.i}}{\boldsymbol{U}^T_{M.i}} \end{bmatrix}^T$ is the matrix consisting of the M submatrices $\underset{J_{m.i}\times J_i}{\boldsymbol{U}_{m.i}}$, $m=\overline{1,M}$ situated one under another. Each submatrix $\underset{J_{m.i}\times J_i}{\boldsymbol{U}_{m.i}}$ is formed from the m-th ($m=\overline{1,M}$) row of the scenario matrix given in Table 1 by splitting it into the $J_{m.i}$ rows, which are all filled with 0 except just one element which is equal to one. Those unit-elements are situated in successive split rows of the submatrices $\underset{J_{m.i}\times J_i}{\boldsymbol{U}_{m.i}}$ at the same places as in the m-th row of the scenario matrix. Figure 1 shows the example of how the submatrix $\underset{J_{1.i}\times J_i}{\boldsymbol{U}_{1.i}}$ is constructed by splitting the 1-st row of the scenario matrix into the $J_{1.i}$=5 rows.

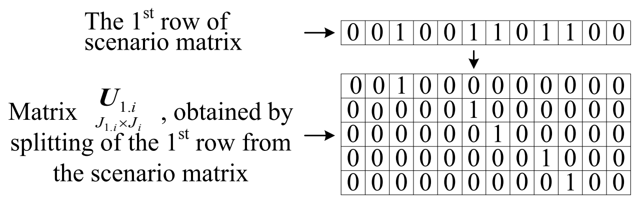

Fig. 1. Splitting the 1-st row of the scenario matrix into $J_{1.i}$=5 rows.

The remaining rows of the scenario matrix are split in the same way; $\underset{3J_{\Sigma.i}\times 2J_{\Sigma.i}}{\boldsymbol{\Lambda}_i} = \begin{bmatrix} \underset{J_{\Sigma.i}\times J_{\Sigma.i}}{\mathbf{0}} & \underset{J_{\Sigma.i}\times J_{\Sigma.i}}{\mathbf{0}} \\ -\lambda_{\Delta nifr}\underset{J_{\Sigma.i}\times J_{\Sigma.i}}{\boldsymbol{E}_i} & -\lambda_{n_2ifr}\underset{J_{\Sigma.i}\times J_{\Sigma.i}}{\boldsymbol{E}_i} \\ \underset{J_{\Sigma.i}\times J_{\Sigma.i}}{\mathbf{0}} & -\lambda_{mw}\underset{J_{\Sigma.i}\times J_{\Sigma.i}}{\boldsymbol{E}_i} \end{bmatrix}$;

$\underset{J_{\Sigma.i}\times J_{\Sigma.i}}{\boldsymbol{E}_i}$ is the identity matrix of dimension $(J_{\Sigma.i}\times J_{\Sigma.i})$.

The matrix $\boldsymbol{V}_{net.i}$ whose columns are the basis vectors of the null space of the design matrix $\underset{3J_{\Sigma.i}\times nx_i}{\boldsymbol{H}_{net.i}}$ (5) was obtained:

$$\underset{nx_i\times(2M+2J_i+1)}{\boldsymbol{V}_{net.i}} = \left[\underset{nx_i\times 3}{\boldsymbol{V1}_{net.i}}\quad \underset{nx_i\times M}{\boldsymbol{V2}_{net.i}}\quad \underset{nx_i\times M}{\boldsymbol{V3}_{net.i}}\quad \underset{nx_i\times(J_i-1)}{\boldsymbol{V4}_{net.i}}\quad \underset{nx_i\times(J_i-1)}{\boldsymbol{V5}_{net.i}}\right] \tag{6}$$

where

$$\underset{nx_i\times 3}{\boldsymbol{V1}_{net.i}} = \begin{bmatrix} \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} \\ \underset{M\times1}{\mathbf{1}} & \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} \\ \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{1}} & \underset{M\times1}{\mathbf{0}} \\ \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{1}} \\ \underset{J_i\times1}{\mathbf{1}} & \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{1}} & \underset{J_i\times1}{\mathbf{0}} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{1}} \\ \underset{2J_{\Sigma.i}\times1}{\mathbf{0}} & \underset{2J_{\Sigma.i}\times1}{\mathbf{0}} & \underset{2J_{\Sigma.i}\times1}{\mathbf{0}} \end{bmatrix}, \quad \underset{nx_i\times M}{\boldsymbol{V2}_{net.i}} = \begin{bmatrix} \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} & \cdots & \underset{M\times1}{\mathbf{0}} \\ \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} & \cdots & \underset{M\times1}{\mathbf{0}} \\ \lambda_{\Delta nifr}\underset{M\times1}{\mathbf{1}^1} & \lambda_{\Delta nifr}\underset{M\times1}{\mathbf{1}^2} & \cdots & \lambda_{\Delta nifr}\underset{M\times1}{\mathbf{1}^M} \\ \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} & \cdots & \underset{M\times1}{\mathbf{0}} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} & \cdots & \underset{J_i\times1}{\mathbf{0}} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} & \cdots & \underset{J_i\times1}{\mathbf{0}} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} & \cdots & \underset{J_i\times1}{\mathbf{0}} \\ \underset{J_{\Sigma.i}\times1}{\mathbf{1}^1} & \underset{J_{\Sigma.i}\times1}{\mathbf{1}^2} & \cdots & \underset{J_{\Sigma.i}\times1}{\mathbf{1}^M} \\ \underset{J_{\Sigma.i}\times1}{\mathbf{0}} & \underset{J_{\Sigma.i}\times1}{\mathbf{0}} & \cdots & \underset{J_{\Sigma.i}\times1}{\mathbf{0}} \end{bmatrix},$$

$$\underset{nx_i\times M}{\boldsymbol{V3}_{net.i}} = \begin{bmatrix} \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} & \cdots & \underset{M\times1}{\mathbf{0}} \\ \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} & \cdots & \underset{M\times1}{\mathbf{0}} \\ \lambda_{n_2ifr}\underset{M\times1}{\mathbf{1}^1} & \lambda_{n_2ifr}\underset{M\times1}{\mathbf{1}^2} & \cdots & \lambda_{n_2ifr}\underset{M\times1}{\mathbf{1}^M} \\ \lambda_{mw}\underset{M\times1}{\mathbf{1}^1} & \lambda_{mw}\underset{M\times1}{\mathbf{1}^2} & \cdots & \lambda_{mw}\underset{M\times1}{\mathbf{1}^M} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} & \cdots & \underset{J_i\times1}{\mathbf{0}} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} & \cdots & \underset{J_i\times1}{\mathbf{0}} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} & \cdots & \underset{J_i\times1}{\mathbf{0}} \\ \underset{J_{\Sigma.i}\times1}{\mathbf{0}} & \underset{J_{\Sigma.i}\times1}{\mathbf{0}} & \cdots & \underset{J_{\Sigma.i}\times1}{\mathbf{0}} \\ \underset{J_{\Sigma.i}\times1}{\mathbf{1}^1} & \underset{J_{\Sigma.i}\times1}{\mathbf{1}^2} & \cdots & \underset{J_{\Sigma.i}\times1}{\mathbf{1}^M} \end{bmatrix},$$

are the matrices, where $\underset{M\times1}{\mathbf{1}^m}$ is the column vector made of M nulls excluding the unit in the m-th, $m=\overline{1,M}$ position; $\underset{J_{\Sigma.i}\times1}{\mathbf{1}^m}$ is the $J_{\Sigma.i}$-vector formed from the M subvectors $\underset{J_{m.i}\times1}{\mathbf{1}^m}$, $m=\overline{1,M}$. All the $J_{m.i}$-subvectors $\underset{J_{m.i}\times1}{\mathbf{1}^m}$ are zeroes, except for the m-th consisting of ones.

$$\underset{nx_i\times(J_i-1)}{\boldsymbol{V4}_{net.i}} = \begin{bmatrix} \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} & \cdots & \underset{M\times1}{\mathbf{0}} \\ \underset{3M\times1}{\mathbf{0}} & \underset{3M\times1}{\mathbf{0}} & \cdots & \underset{3M\times1}{\mathbf{0}} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} & \cdots & \underset{J_i\times1}{\mathbf{0}} \\ -\lambda_{\Delta nifr}\underset{J_i\times1}{\mathbf{1}^1} & -\lambda_{\Delta nifr}\underset{J_i\times1}{\mathbf{1}^2} & \cdots & -\lambda_{\Delta nifr}\underset{J_i\times1}{\mathbf{1}^{J_i-1}} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} & \cdots & \underset{J_i\times1}{\mathbf{0}} \\ \underset{J_{1.i}\times1}{\mathbf{1s}_1^1} & \underset{J_{1.i}\times1}{\mathbf{1s}_1^2} & \cdots & \underset{J_{1.i}\times1}{\mathbf{1s}_1^{J_i-1}} \\ \underset{J_{2.i}\times1}{\mathbf{1s}_2^1} & \underset{J_{2.i}\times1}{\mathbf{1s}_2^2} & \cdots & \underset{J_{2.i}\times1}{\mathbf{1s}_2^{J_i-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \underset{J_{M.i}\times1}{\mathbf{1s}_M^1} & \underset{J_{M.i}\times1}{\mathbf{1s}_M^2} & \cdots & \underset{J_{M.i}\times1}{\mathbf{1s}_M^{J_i-1}} \\ \underset{J_{\Sigma.i}\times1}{\mathbf{0}} & \underset{J_{\Sigma.i}\times1}{\mathbf{0}} & \cdots & \underset{J_{\Sigma.i}\times1}{\mathbf{0}} \end{bmatrix}$$

$$\underset{nx_i\times(J_i-1)}{\boldsymbol{V5}_{net.i}} = \begin{bmatrix} \underset{M\times1}{\mathbf{0}} & \underset{M\times1}{\mathbf{0}} & \cdots & \underset{M\times1}{\mathbf{0}} \\ \underset{3M\times1}{\mathbf{0}} & \underset{3M\times1}{\mathbf{0}} & \cdots & \underset{3M\times1}{\mathbf{0}} \\ \underset{J_i\times1}{\mathbf{0}} & \underset{J_i\times1}{\mathbf{0}} & \cdots & \underset{J_i\times1}{\mathbf{0}} \\ \underset{J_k\times1}{\mathbf{0}} & \underset{J_k\times1}{\mathbf{0}} & \cdots & \underset{J_k\times1}{\mathbf{0}} \\ \lambda_{mw}\underset{J_i\times1}{\mathbf{1}^1} & \lambda_{mw}\underset{J_i\times1}{\mathbf{1}^2} & \cdots & \lambda_{mw}\underset{J_i\times1}{\mathbf{1}^{J_i-1}} \\ -\frac{\lambda_{n_2ifr}}{\lambda_{\Delta nifr}}\underset{J_{1.i}\times1}{\mathbf{1s}_1^1} & -\frac{\lambda_{n_2ifr}}{\lambda_{\Delta nifr}}\underset{J_{1.i}\times1}{\mathbf{1s}_1^2} & \cdots & -\frac{\lambda_{n_2ifr}}{\lambda_{\Delta nifr}}\underset{J_{1.i}\times1}{\mathbf{1s}_1^{J_i-1}} \\ -\frac{\lambda_{n_2ifr}}{\lambda_{\Delta nifr}}\underset{J_{2.i}\times1}{\mathbf{1s}_2^1} & -\frac{\lambda_{n_2ifr}}{\lambda_{\Delta nifr}}\underset{J_{2.i}\times1}{\mathbf{1s}_2^2} & \cdots & -\frac{\lambda_{n_2ifr}}{\lambda_{\Delta nifr}}\underset{J_{2.i}\times1}{\mathbf{1s}_2^{J_i-1}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\lambda_{n_2ifr}}{\lambda_{\Delta nifr}}\underset{J_{M.i}\times1}{\mathbf{1s}_M^1} & -\frac{\lambda_{n_2ifr}}{\lambda_{\Delta nifr}}\underset{J_{M.i}\times1}{\mathbf{1s}_M^2} & \cdots & -\frac{\lambda_{n_2ifr}}{\lambda_{\Delta nifr}}\underset{J_{M.i}\times1}{\mathbf{1s}_M^{J_i-1}} \\ \underset{J_{1.i}\times1}{\mathbf{1s}_1^1} & \underset{J_{1.i}\times1}{\mathbf{1s}_1^2} & \cdots & \underset{J_{1.i}\times1}{\mathbf{1s}_1^{J_i-1}} \\ \underset{J_{2.i}\times1}{\mathbf{1s}_2^1} & \underset{J_{2.i}\times1}{\mathbf{1s}_2^2} & \cdots & \underset{J_{2.i}\times1}{\mathbf{1s}_2^{J_i-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \underset{J_{M.i}\times1}{\mathbf{1s}_M^1} & \underset{J_{M.i}\times1}{\mathbf{1s}_M^2} & \cdots & \underset{J_{M.i}\times1}{\mathbf{1s}_M^{J_i-1}} \end{bmatrix}$$

are the matrices, in which $\underset{J_{m.i}\times1}{\mathbf{1s}_{m.i}^j}$ $j=\overline{1,J_i-1}$, $m=\overline{1,M}$ are the column vectors of the submatrices $\underset{J_{m.i}\times J_i}{\boldsymbol{U}_{m.i}}$ (excluding the last column vector) of the matrix $\underset{J_{\Sigma.i}\times J_i}{\boldsymbol{U}_i}$, the construction of which was described earlier. For example, the $J_{1,i}$-1 column vectors $\left[\underset{J_{1.i}\times1}{\boldsymbol{1s}_1^1}\quad \underset{J_{1.i}\times1}{\boldsymbol{1s}_1^2}\quad \cdots\quad \underset{J_{1.i}\times1}{\boldsymbol{1s}_1^{J_i-1}}\right]$ form the matrix $\underset{J_{1.i}\times J_i}{\boldsymbol{U}_{1.i}}$ without the last column. This matrix is shown in Fig. 1, where $J_{1.i}=5$ and $J_i=12$.

The number of columns in the matrix $\underset{nx_i\times(2M+2J_i+1)}{\boldsymbol{V}_{net.i}}$ (6) is $2M+2J_i+1$ and, therefore, the rank deficiency of the design matrix $\underset{3J_{\Sigma.i}\times nx_i}{\boldsymbol{H}_{net.i}}$ (5) is $dfh_i=2M+2J_i+1$, and its rank is $rnkh_i=nx_i-dfhk=2M+J_i+2J_{\Sigma.i}-1$. It can be seen that in the network solution, by contrast to the user solution, the rank deficiency dfh_i of the matrix $\underset{3J_{\Sigma.i}\times nx_i}{\boldsymbol{H}_{net.i}}$ (5) depends on the number J_i of observed satellites and the number of the stations M.

As in the user solution, the linear system (3) is singular, i.e., it has an infinite set of solutions lying in the dfh_i-dimensional solution space parallel-shifted with respect to the null-space $\underset{nx_i\times(2M+2J_i+1)}{\boldsymbol{V}_{net.i}}$ (6). However, as can be seen from (6), the first M elements of the null-space basis vectors $\underset{nx_i\times(2M+2J_i+1)}{\boldsymbol{V}_{net.i}}$ (6) are zeroes. This means that the set of solutions of the system (3) is orthogonal to those axes of the space variables, which correspond to the first M elements of the vector $\underset{nx_i\times1}{\boldsymbol{x}_{net.i}}$ (4). As it can be seen from (4), these elements compose the vector $\underset{M\times1}{\boldsymbol{\Delta D}_i}$. Thus, the first M coordinates of the points lying in the dfh_i-dimensional solution space are the same for all points of this space and, hence, all M elements of the vector $\underset{M\times1}{\boldsymbol{\Delta D}_i}$ can be estimated unambiguously. The remaining elements of the state vector $\underset{nx_i\times1}{\boldsymbol{x}_{net.i}}$ (4) are co-variables, i.e., they cannot be estimated per se, but only some linear combinations thereof are estimable. The co-variables include the elements of the integer vectors $\underset{J_{\Sigma.i}\times1}{\boldsymbol{N}1}$, $\underset{J_{\Sigma.i}\times1}{\boldsymbol{N}_{mw}}$, as well as the vectors $\underset{J_i\times1}{\boldsymbol{dt}_{\rho.i}}$, $\underset{J_i\times1}{\boldsymbol{dt}_{L.i}}$, $\underset{J_i\times1}{\boldsymbol{b}_{mw.i}}$, estimation of which is the purpose of the network solution. Now we are going to derive the expressions for linear combinations, which include these vectors.

As in the case of a user solution, the determination of linear combinations formed by co-variables is performed with the help of the S-transformation theory [2–5], i.e., by projecting all the points of the space of the variables onto the S-subspace along the null-space $\underset{nx_i\times(2M+2J_i+1)}{\boldsymbol{V}_{net.i}}$ (6). The dimension of the S-subspace equals the rank of the matrix $\underset{3J_{\Sigma.i}\times nx_i}{\boldsymbol{H}_{net.i}}$ (5) of the linear system (3). Similar to a user solution, the S-subspace of the network solution can be obtained from the system of normal equations [2, 3]

$$\Big(\underset{dfh_i\times nx_i}{\boldsymbol{S}^{\perp}_{net.i}}\Big)^T \underset{nx_i\times1}{\boldsymbol{x}_{net.i}} = \underset{dfh_i\times1}{\boldsymbol{0}} \tag{7}$$

where $\underset{nx_i\times dfh_i}{\boldsymbol{S}^{\perp}_{net.i}}$ is the matrix of the rank dfh_i, and all the dfh_i column vectors are orthogonal to the S-subspace. By projecting point coordinates, we get a new vector of estimated variables $\underset{nx_i\times1}{\boldsymbol{x}_{net.s.i}}$ of the same dimension as the original vector $\underset{nx_i\times1}{\boldsymbol{x}_{net.i}}$ (4). The relation of vectors $\underset{nx_i\times1}{\boldsymbol{x}_{net.s.i}}$, $\underset{nx_i\times1}{\boldsymbol{x}_{net.i}}$ is defined by the expression

$$\underset{nx_i\times1}{\boldsymbol{x}_{net.s.i}} = \underset{nx_i\times nx_i}{\boldsymbol{P}_{net.i}}\ \underset{nx_i\times1}{\boldsymbol{x}_{net.i}} \tag{8}$$

where $\underset{nx_i\times nx_i}{\boldsymbol{P}_{net.i}} = \underset{nx_i\times nx_i}{\boldsymbol{E}_{i}} - \underset{nx_i\times dfh_i}{\boldsymbol{V}_{net.i}}\left(\Big(\underset{dfh_i\times nx_i}{\boldsymbol{S}^{\perp}_{net.i}}\Big)^T \underset{nx_i\times dfh_i}{\boldsymbol{V}_{net.i}}\right)^{-1}\Big(\underset{dfh_i\times nx_i}{\boldsymbol{S}^{\perp}_{net.i}}\Big)^T$. To preserve the integer nature of linear combinations of integer co-variables, which are the components of the integer vectors $\underset{J_{\Sigma.i}\times1}{\boldsymbol{N}1}$ and $\underset{J_{\Sigma.i}\times1}{\boldsymbol{N}_{mw}}$ included in the initial state vector $\underset{nx_i\times1}{\boldsymbol{x}_{net.i}}$ (4), the columns of the matrix $\underset{nx_i\times dfh_i}{\boldsymbol{S}^{\perp}_{net.i}}$ should be set in the same way as in the user solution [1]. Namely, all the elements in the columns of the matrix $\underset{nx_i\times dfh_i}{\boldsymbol{S}^{\perp}_{net.i}}$ should be set to zero, excluding the only element equal to 1.

The estimate $\underset{nx_i\times1}{\hat{\boldsymbol{x}}_{net.s.i}}$ of a new state vector $\underset{nx_i\times1}{\boldsymbol{x}_{net.s.i}}$ in the network solution can be found from the solution of the extended system of linear equations obtained by combining the linear systems (3) and (7)

$$\begin{bmatrix} \underset{3J_{\Sigma.i}\times1}{\boldsymbol{Y}_{net.i}} \\ \underset{dfh_i\times1}{\boldsymbol{0}} \end{bmatrix} = \begin{bmatrix} \underset{3J_{\Sigma.i}\times nx_i}{\boldsymbol{H}_{net.i}} \\ \Big(\underset{dfh_i\times nx_i}{\boldsymbol{S}^{\perp}_{net.i}}\Big)^T \end{bmatrix} \cdot \underset{nx_i\times1}{\boldsymbol{x}_{net.s.i}} + \begin{bmatrix} \underset{3J_{\Sigma.i}\times1}{\boldsymbol{\Xi}_{net.i}} \\ \underset{dfh_i\times1}{\boldsymbol{0}} \end{bmatrix}. \tag{9}$$

From the equation $\Big(\underset{dfh_i\times nx_i}{\boldsymbol{S}^{\perp}_{net.i}}\Big)^T \underset{nx_i\times1}{\boldsymbol{x}_{net.i}} = \underset{dfh_i\times1}{\boldsymbol{0}}$ (7), taking into account the specific type of the columns of the matrix $\underset{nx_i\times dfh_i}{\boldsymbol{S}^{\perp}_{net.i}}$, it follows that the elements of the solution vector $\underset{nx_i\times1}{\hat{\boldsymbol{x}}_{net.s.i}}$ of the system (9) standing on the places defined by the position of the units in the dfh_i columns of the matrix $\underset{nx_i\times dfh_i}{\boldsymbol{S}^{\perp}_{net.i}}$ equal zero. But if it is known in advance that dfh_i of the elements of the solution vector $\underset{nx_i\times1}{\hat{\boldsymbol{x}}_{net.s.i}}$ of the system (9) are zero, hence, the estimates of the remaining $rnkh_i$ elements of the solution vector of the system (3) can be obtained by solving a simpler system of the linear equations

$$\underset{3J_{\Sigma.i}\times1}{\boldsymbol{Y}_{net.i}} = \underset{3J_{\Sigma.i}\times(nx_i-dfh_i)}{\boldsymbol{H}_{net.cmpr.i}}\ \underset{(nx_i-dfh_i)\times1}{\boldsymbol{x}_{net.cmpr.i}} + \underset{3J_{\Sigma.i}\times1}{\boldsymbol{\Xi}_{i}} \tag{10}$$

where $\underset{3J_{\Sigma.i}\times(nx_i-dfh_i)}{\boldsymbol{H}_{net.cmpr.i}}$ is the compressed form of the initial matrix $\underset{3J_{\Sigma.i}\times nx_i}{\boldsymbol{H}_{net.i}}$ (5), where the dfh_i columns are cut from the positions in which the units are located in the dfh_i columns of the matrix $\underset{nx_i\times dfh_i}{\boldsymbol{S}^{\perp}_{net.i}}$; $\underset{(nx_i-dfh_i)\times1}{\boldsymbol{x}_{net.cmpr.i}}$ is the compressed state vector obtained from the initial state vector $\underset{nx_i\times 1}{\boldsymbol{x}_{net.s.i}}$, where all the zeroes are excluded. Hence, we see that positions of ones in dfh_i columns of the matrix $\underset{nx_i\times dfh_i}{\boldsymbol{S}^{\perp}_{net.i}}$ determine the positions of excluded columns dfh_i of the initial matrix $\underset{3J_{\Sigma.i}\times nx_i}{\boldsymbol{H}_{net.i}}$ (5). Nevertheless, this manipulation of dropping the columns makes the rank of the compressed matrix $\underset{3J_{\Sigma.i}\times(nx_i-dfh_i)}{\boldsymbol{H}_{net.cmpr.i}}$ match the rank $rnkh_i$ of the initial matrix $\underset{3J_{\Sigma.i}\times nx_i}{\boldsymbol{H}_{net.i}}$ (5). This requirement is satisfied if one of the M ground stations with the number r (reference) $1\le r\le M$ is defined as reference and if the matrix $\underset{nx_i\times dfh_i}{\boldsymbol{S}^{\perp}_{net.i}}$ is constructed by merging three submatrices

$$\underset{nx_i\times dfh_i}{\boldsymbol{S}^{\perp}_{net.i}}=\left[\underset{nx_i\times 3}{\boldsymbol{S}^{\perp}_{net.time.i}}\quad \underset{nx_i\times(dfh_i-3)/2}{\boldsymbol{S}^{\perp}_{net.N1.i}}\quad \underset{nx_i\times(dfh_i-3)/2}{\boldsymbol{S}^{\perp}_{net.N_{mw}.i}}\right].$$

The units in three columns of the first submatrix $\underset{nx_i\times 3}{\boldsymbol{S}^{\perp}_{net.time.i}}$ are located on the positions $M+r$, $2M+r$, $3M+r$, respectively. Placing the units in the columns of the submatrix $\underset{nx_i\times 3}{\boldsymbol{S}^{\perp}_{net.time.i}}$ this way, we posit them in the same positions as the ionosphere-free code $dT_{\rho.r.i}$ and phase $dT_{L.r.i}$ clock offsets and the biases of the Melbourne-Wübbena combination $b_{mw.r}$ for the r-th ground station in the initial state vector $\underset{nx_i\times 1}{\boldsymbol{x}_{net.i}}$. The location of units in $(dfh_i-3)/2=M+J_i-1$ columns of the submatrices $\underset{nx_i\times(M+J_i-1)}{\boldsymbol{S}^{\perp}_{net.N1.i}}$, $\underset{nx_i\times(M+J_i-1)}{\boldsymbol{S}^{\perp}_{net.N_{mw}.i}}$ is determined by position of units in the spanning tree matrix $\underset{M\times J_i}{\boldsymbol{STM}}$ elements of which are either zero or one. The matrix $\underset{M\times J_i}{\boldsymbol{STM}}$ has the same dimension $M\times J_i$ as the scenario matrix, while the number of units in $\underset{M\times J_i}{\boldsymbol{STM}}$ equals $M+J_i-1$, and all these units constitute the unit-subspace of the scenario matrix. Hence, $\underset{M\times J_i}{\boldsymbol{STM}}$ can be shown by highlighting in grey some units in the scenario matrix. An example of such highlighting is shown in Table 1.

$\underset{M\times J_i}{\boldsymbol{STM}}$ is obtained from the scenario matrix through a special algorithm, which can be found in [2, 6–9]. In particular, Prim's algorithm applicable to weighted non-directed graphs is briefly described as follows [2]:

1. Take the edge with the highest weight and place two thereby connected nodes in the set V_1;

2. Take the set of edges that connect nodes from $V-V_1$ to nodes from V_1, and select the edge with the highest weight;

3. Add the node belonging to $V-V_1$ of the edges selected in step 2 to V_1;

4. While $V-V_1\neq 0$, go to step 2.

The number of units in $\underset{M\times J_i}{\boldsymbol{STM}}$, i.e., $M+J_i-1$, equals the number of columns of submatrices $\underset{nx_i\times(M+J_i-1)}{\boldsymbol{S}^{\perp}_{net.N1.i}}$, $\underset{nx_i\times(M+J_i-1)}{\boldsymbol{S}^{\perp}_{net.N_{mw}.i}}$, i.e., each unit of $\underset{M\times J_i}{\boldsymbol{STM}}$ is placed in a certain way within the rows numbered $l=\overline{4M+3J_i+1,\ \ 4M+3J_i+J_{\Sigma.i}}$ of one of the columns of the submatrix $\underset{nx_i\times(M+J_i-1)}{\boldsymbol{S}^{\perp}_{net.N1.i}}$ and in the rows with the numbers $l=\overline{4M+3J_i+J_{\Sigma.i}+1,\ \ 4M+3J_i+2J_{\Sigma.i}}$ of one of the columns of the submatrix $\underset{nx_i\times(M+J_i-1)}{\boldsymbol{S}^{\perp}_{net.N_{mw}.i}}$ according to the following algorithm. All the units comprising the scenario matrix are indexed by the first lower index in the order of their location from the left to the right and for rows from up to down in the range from 1 to $J_{\Sigma.i}$. Then, the units of the scenario matrix, which comprise the matrix $\underset{M\times J_i}{\boldsymbol{STM}}$, are indexed by the second lower index in the same order in the range from 1 to $M+J_i-1$. As a result, units comprising the matrix $\underset{M\times J_i}{\boldsymbol{STM}}$, will have two indices, and the remaining units will have only one. An example of such indexation is shown in Table 1. Dual-indexed units comprising the matrix $\underset{M\times J_i}{\boldsymbol{STM}}$ are highlighted in grey. Let us denote the first index of dual-indexed units as μ ($\mu=\overline{1, J_{\Sigma.i}}$), while the second index is ν ($\nu=\overline{1, M+J_i-1}$). Each dual-indexed unit corresponds to the column ν of the matrices $\underset{nx_i\times(M+J_i-1)}{\boldsymbol{S}^{\perp}_{net.N1.i}}$ and $\underset{nx_i\times(M+J_i-1)}{\boldsymbol{S}^{\perp}_{net.N_{mw}.i}}$. The element $4M+3J_i+\mu$ of ν-th column of the matrix $\underset{nx_i\times(M+J_i-1)}{\boldsymbol{S}^{\perp}_{net.N1.i}}$ is replaced by one, and the remaining elements of this column are zero. Similarly, the element $4M+3J_i+J_{\Sigma.i}+\mu$ of ν-th column of the matrix $\underset{nx_i\times(M+J_i-1)}{\boldsymbol{S}^{\perp}_{net.N_{mw}.i}}$ is replaced by one, while the remaining elements of this column are zeroes.

A unique solution of the system (10) can be obtained if the number of rows of the compressed matrix $\underset{3J_{\Sigma.i}\times(nx_i-dfh_i)}{\boldsymbol{H}_{net.cmpr.i}}$

is greater or equal to the dimension of the compressed state vector $\underset{3J_{\Sigma.i}\times(nx_i-dfh_i)}{\boldsymbol{H}_{net.cmpr.i}}$, i.e., the condition $3J_{\Sigma.i} \geq (nx_i - dfh_i) = 2M+J_i+2J_{\Sigma.i}-1$ should be fulfilled. Hence, we obtain the following constraint $J_{\Sigma.i} \geq 2M+J_i-1$, which should be met to get a unique network solution.

In general, the analytical calculation of the vector $\underset{nx_i\times1}{\boldsymbol{x}_{net.s.i}}$ by the formula (8), which will include linear combinations of elements of the vectors $\underset{J_{\Sigma.i}\times1}{\boldsymbol{N}1}$, $\underset{J_{\Sigma.i}\times1}{\boldsymbol{N}_{mw}}$, as well as the vectors $\underset{J_i\times1}{\boldsymbol{dt}_{\rho.i}}$, $\underset{J_i\times1}{\boldsymbol{dt}_{L.i}}$ and $\underset{J_i\times1}{\boldsymbol{b}_{mw.i}}$, is cumbersome. To reduce the complexity, let us consider the computation of the vector $\underset{nx_i\times1}{\boldsymbol{x}_{net.s.i}}$ for an extremely simplified case when M=3, J_i=4, for which the scenario matrix and $\underset{M\times J_i}{\boldsymbol{STM}}$ are given in Table 2.

Table 2. SV visibility by the ground stations for the simplified case with M=3, J_i=4, $J_{\Sigma.i}$=9

	m	Numbers of SVs visible by ground stations				J_m
		1	2	3	4	
Ground network stations	1	1_{11}	1_{22}	1_{33}	0	3
	2	1_{44}	0	1_{5}	1_{65}	3
	3	0	1_{7}	1_{8}	1_{96}	3

For the case M=3, J_i=4, $J_{\Sigma.i}$=9, the vector of the variables $\underset{42\times1}{\boldsymbol{x}_{net.s.i}}$ (8) and the corresponding projection matrix $\underset{42\times42}{\boldsymbol{P}_{net.k}}$ will be quite bulky even in the simplified case. Hence, the expressions below are presented only for the subvectors of interest $\underset{M\times1}{\boldsymbol{\Delta D}_{s.i}}$, $\underset{4\times1}{\boldsymbol{dt}_{\rho.s.i}}$, $\underset{4\times1}{\boldsymbol{dt}_{L.s.i}}$, $\underset{4\times1}{\boldsymbol{b}_{mw.s.i}}$, $\underset{9\times1}{\boldsymbol{N1}_{s.i}}$, $\underset{9\times1}{\boldsymbol{N}_{mw.s.i}}$ of the vector $\underset{42\times1}{\boldsymbol{x}_{net.s.i}}$ (8) obtained by symbolic computations in MATLAB for r=1.

$$\underset{M\times1}{\boldsymbol{\Delta D}_{i.k}} = \begin{bmatrix} \Delta D_{1.i} \\ \Delta D_{2.i} \\ \Delta D_{3.i} \end{bmatrix},$$

$$\underset{4\times1}{\boldsymbol{dt}_{\rho.s.i}} = \begin{bmatrix} dt^1_{\rho.i} - dT_{\rho.r.i} \\ dt^2_{\rho.i} - dT_{\rho.r.i} \\ dt^3_{\rho.i} - dT_{\rho.r.i} \\ dt^4_{\rho.i} - dT_{\rho.r.i} \end{bmatrix},$$

$$\underset{4\times1}{\boldsymbol{dt}_{L.s.i}} = \begin{bmatrix} dt^1_{L.i} - dT_{L.r.i} + \lambda_{\Delta nifr} N1^1_1 + \lambda_{n2ifr} N^1_{mw1} \\ dt^2_{L.i} - dT_{L.r.i} + \lambda_{\Delta nifr} N1^2_1 + \lambda_{n2ifr} N^2_{mw1} \\ dt^3_{L.i} - dT_{L.r.i} + \lambda_{\Delta nifr} N1^3_1 + \lambda_{n2ifr} N^3_{mw1} \\ dt^4_{L.i} - dT_{L.r.i} + \lambda_{\Delta nifr}\left(N1^1_1 - N^1_2 + N^4_2\right) + \lambda_{n2ifr}\left(N^1_{mw1} - N^1_{mw2} + N^4_{mw2}\right) \end{bmatrix},$$

$$\underset{4\times1}{\boldsymbol{b}_{mw.s.i}} = \begin{bmatrix} b^1_{mw} - b_{mw.r} + \lambda_{mw} N^1_{mw.1} \\ b^2_{mw} - b_{mw.r} + \lambda_{mw} N^2_{mw.1} \\ b^3_{mw} - b_{mw.r} + \lambda_{mw} N^3_{mw.1} \\ b^4_{mw} - b_{mw.r} + \lambda_{mw}\left(N^1_{mw.1} - N^1_{mw.2} + N^4_{mw.2}\right) \end{bmatrix},$$

$$\underset{9\times1}{\boldsymbol{N1}_{s.i}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ N1^3_2 - N1^1_2 + N1^1_1 - N1^3_1 \\ 0 \\ N1^2_3 - N1^4_3 + N1^4_2 - N1^1_2 + N1^1_1 - N1^2_1 \\ N1^3_3 - N1^4_3 + N1^4_2 - N1^1_2 + N1^1_1 - N1^3_1 \\ 0 \end{bmatrix},$$

$$\underset{9\times1}{\boldsymbol{N}_{mw.s.i}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ N1^3_{mw2} - N1^1_{mw2} + N1^1_{mw1} - N1^3_{mw1} \\ 0 \\ N1^2_{mw3} - N1^4_{mw3} + N1^4_{mw2} - N1^1_{mw2} + N1^1_{mw1} - N1^2_{mw1} \\ N1^3_{mw3} - N1^4_{mw3} + N1^4_{mw2} - N1^1_{mw2} + N1^1_{mw1} - N1^3_{mw1} \\ 0 \end{bmatrix}. \qquad (11)$$

As can be seen from the expressions (11), the elements of the vector $\underset{M\times1}{\boldsymbol{\Delta D}^T} = \left[\Delta D_{1.i} \quad \Delta D_{2.i} \quad \Delta D_{3.i}\right]^T$ of uncompensated wet component of the zenith tropospheric delays (m) at the locations of three ground stations, as expected, are unbiased. The initial integer ambiguities $\underset{9\times1}{\boldsymbol{N1}_i}$ and $\underset{9\times1}{\boldsymbol{N}_{mw.i}}$ are estimated with biases, i.e., as a part of the linear combinations $\underset{9\times1}{\boldsymbol{N1}_{s.i}}$ and $\underset{9\times1}{\boldsymbol{N}_{mw.s.i}}$, which are also integer. We are interested in the vectors $\underset{4\times1}{\boldsymbol{dt}_{\rho.i}}$, $\underset{4\times1}{\boldsymbol{dt}_{L.i}}$, and $\underset{4\times1}{\boldsymbol{b}_{mw.i}}$ which are also estimated with biases, i.e., are composed of the linear combinations $\underset{4\times1}{\boldsymbol{dt}_{\rho.s.i}}$, $\underset{4\times1}{\boldsymbol{dt}_{L.s.i}}$, and $\underset{4\times1}{\boldsymbol{b}_{mw.s.i}}$. However, as it is shown in (11), for all J_i SVs, the estimation biases of the variables included in the vector $\underset{4\times1}{\boldsymbol{dt}_{\rho.i}}$ are the same and equal to ionosphere-free code clock offsets $dT_{\rho.r.i}$ of the reference station; the estimation biases of the variables included in the vector $\underset{4\times1}{\boldsymbol{dt}_{L.i}}$ are accurate within an integer combination of the wavelengths $\lambda_{\Delta nifr}$, $\lambda_{n_2 ifr}$, and the estimation biases of the variables included in the vector $\underset{4\times1}{\boldsymbol{b}_{mw.i}}$ are accurate within an integer number of the wavelengths λ_{mw}. It leads to respective biases in the residuals of the ionosphere-free code $\Delta\rho^j_{ifr.i} = \rho^j_{ifr.i} - R^j_{c.i} + dt^j_{\rho.ifr.i}$, carrier phase $\Delta L^j_{ifr.i} = L^j_{ifr.i} - R^j_{c.i} + dt^j_{L.ifr.i}$ and Melbourne-Wübbena combinations $\Delta mw^j_i = mw^j_i + b^j_{mw.i}$ measurements included in the left part of the system of equations in the user solution. However, the properties of this system are such that these biases in the left part do not change the estimates of the unambiguously estimated variables Δx, Δy, Δz, ΔD_i. Thus, to obtain estimates of the variables Δx, Δy, Δz, ΔD_i in the user solution instead of estimates of variables which are components of the vectors $\underset{J_i\times1}{\boldsymbol{dt}_{\rho.i}}$, $\underset{J_i\times1}{\boldsymbol{dt}_{L.i}}$, $\underset{J_i}{\boldsymbol{b}_{mw.i}}$, one can use their biased equivalents, which are components of the vectors $\underset{J_i\times1}{\boldsymbol{dt}_{\rho.s.i}}$, $\underset{J_i\times1}{\boldsymbol{dt}_{L.s.i}}$, $\underset{J_i\times1}{\boldsymbol{b}_{mw.s.i}}$ (11).

The algorithms of solving the linear equation system (10) taking into account the integer nature of the part of its variables, are the basis of the algorithms of estimating the variables which are the part of the vectors $\underset{J_i\times1}{\boldsymbol{dt}_{\rho.s.i}}$, $\underset{J_i\times1}{\boldsymbol{dt}_{L.s.i}}$ and $\underset{J_i\times1}{\boldsymbol{b}_{mw.s.i}}$, which are the aim of solving the network solution of the Integer PPP. Unfortunately, the restrictions for the article's volume do not allow us to explore these algorithms in this paper. We can only refer a reader to existing literature on the methods of linear recurrent estimating [10, 11] and on phase integer ambiguity resolution [12–15].

Examples of determination of precise corrections and their features

Two versions of the network solution were implemented. The first solution was obtained with using 5 European stations assuming the permanence of the SV constellation (all stations of the ground network receive measurements from the same set of 6 SVs). Figure 2 shows decoupled code and carrier phase satellite corrections calculated for one of the 6 satellites in the first version of the network solution.

According to (11), the bias between code and phase corrections shown in Fig. 2 may differ from the true one by an integer number of the wavelengths $\lambda_{\Delta nifr}$, $\lambda_{n_2 ifr}$. It can be seen that this bias is constant during the permanent observation scenario.

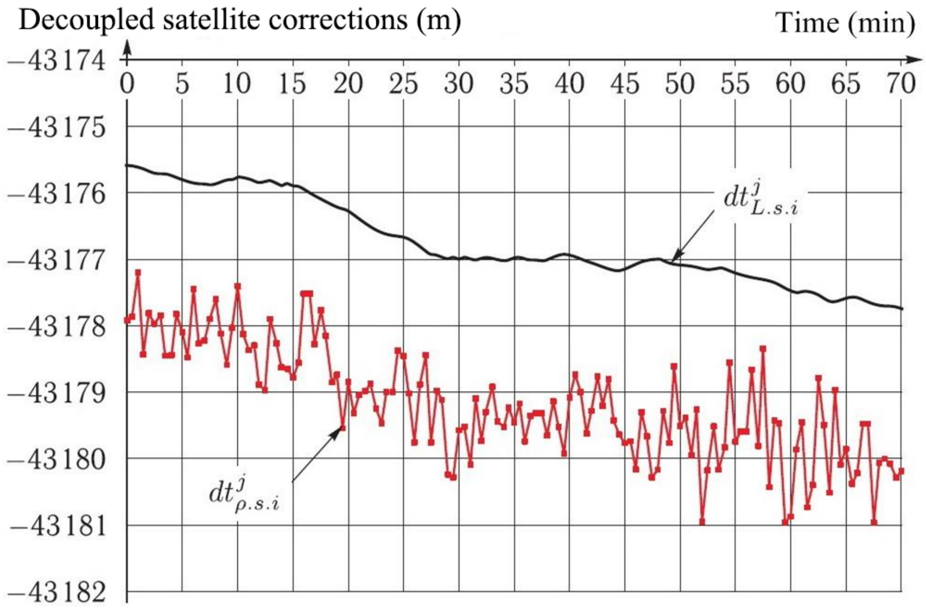

Fig. 2. Decoupled satellite corrections (code and phase) for one of 6 SVs calculated for the first version of the network solution.

Fig. 3. SDCM network stations used in the second version of a network solution.

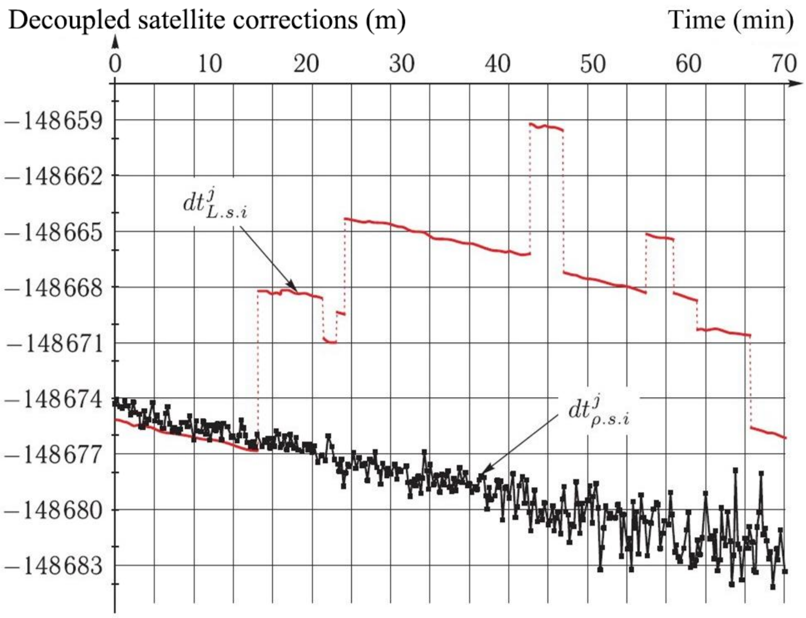

Fig. 4. Decoupled satellite corrections (code and phase) for one of the SVs calculated in the second version of a network solution.

Within the second version of a network solution, measurements from 10 stations of Russian SBAS (SDCM) network were used (highlighted in Fig. 3 by green circles), taking into account changing satellite constellation. The receivers installed at different stations of the network had dissimilar characteristics; hence, the accuracy of the estimation of decoupled satellite corrections was somewhat degraded which caused overall reduction of the accuracy of the user solution.

Figure 4 [16] shows the decoupled code and carrier phase satellite corrections calculated for one of the satellites in the second version of the network solution. The graph shows jumps in phase corrections dt_L^j at the moments of change in the observation scenario and/or change of the matrix $\underset{M\times J_i}{\boldsymbol{STM}}$ associated with changes in the estimated linear combinations in the components of the vector $\underset{J_i\times 1}{\boldsymbol{dt}_{L}}$.

The results of the user solution applying the evaluated corrections shown in Figs. 2 and 4 were already presented and discussed in [1].

Conclusion

Algebraic principles of network solutions for PPP including ambiguity resolution of carrier phase measurements in GNSS with CDMA are considered.

Examples of determination of precise satellite corrections for the GPS Integer PPP are presented. Significant reduction in convergence time required to achieve high-precision positioning when using precise satellite corrections in comparison with the Float PPP is demonstrated.

As it follows from the expression for the vector $\underset{4\times1}{\boldsymbol{dt}_{\rho.s.i}}$ that is included into (11), in the case of a timely determination of the ionosphere-free code clock offset $dT_{\rho.r.i}$ of a reference ground station relative to the GNSS time scale, there appears a possibility of rapid evaluation of the ionosphere-free code clocks offsets $dt^{j}_{\rho.i}$ $j=\overline{1, J_i}$ of all the J_i SVs which are visible by ground stations at the i-th epoch. This data can be used to increase the accuracy of broadcast clock corrections transmitted in the SV navigation messages.

The experience gained so far shows the urgent need for the methods of receiver calibration to deal with the algorithms of determination of precise decoupled satellite corrections.

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