ROCKET-SPACE DEVICE ENGINEERING AND INFORMATION SYSTEMS
Vol. 5, Iss. 1, 2018, pp. 71–75

Sampling Theorem and Practical Usage of Integer Functions for Signal Representation on the Receiving Side

A.I. Troshchenkov, *post-graduate student, as181191@mail.ru*
Joint Stock Company "Russian Space Systems", Moscow, Russian Federation

Abstract. The article contains the provisions uniting engineering and mathematical approaches to process telemetry signals and other information signals on the receiving side. The connection between a tabular composition of the values of telemetric parameters and the functions simulating these parameters is shown. The method to evaluate the nature of the change of different types of telemetry parameters is described. A number of the tasks, the solution of which will make possible to apply the mathematical apparatus of Kotelnikov for economical coding of information as an effective alternative to the methods used today, is offered.
The pay-offs of the paper consist in clearly set of engineering tasks, innovative approaches to processing and representation of the space telemetry. In conjunction with application of modern mathematical methods of data transformation, the presented provisions allow one to realize the development of essentially new methods of information transfer.

Keywords: sampling theorem, continuous function, telemetry signal, spectral form, cutoff frequency, reading, sample, sample rate, time axis, approximating polynomial, orthogonal basis

In radio communication, the sampling theorem is known [1–3]. In the formulation of C. Shannon [4], the theorem is as follows:

Theorem. *If the f (x) function does not contain components with a frequency over W Hz, then it is completely defined by the sequence of its values in the points remote at distance 1/2 W from each other.*

Shannon called the theorem "the sampling theorem". Under this name it is also known in English-speaking technical literature.

Further in the paper the terms "function" and "signal" will be considered synonyms, and *f(x) = f(t).*

Definition. The functions, which do not contain components of frequencies above the set frequencies (higher than W Hz), are called functions with a limited spectrum.

A statement formulated in the theorem is the sampling theorem in time representation. Also the sampling theorem in frequency representation is known (for example, [4]).

Definition. "Sampling" in the Shannon's theorem is the sequence of its values in the points remote at distance 1/2 W from each other. Actually, based on these values, support functions which sum represents a restored form of a signal are formed.

It is known that any continuous function can be spread out on a final piece in a Fourier series that is presented in a spectral form – in the form of the sum of a number of sinusoids with multiple (enumerated) frequencies with certain amplitudes and phases. At rather smooth functions the spectrum quickly decreases (spectrum module coefficients quickly tend to zero). For representation of the "cut-up" functions, with gaps and "breaks", the sinusoids with greater frequencies are necessary.

In engineering practice, nobody remembers that Shannon has behind "sampling" a function, and engineers do not use these support functions to restore a signal. In engineering, "sampling" is understood not as a function, but the value of a signal in the set time of "sample" of a signal ("the sequence of its values in the points remote at distance 1/2 W from each other").

Definition. "Sample" is a process of determination of the current instant value of a signal in the set time point.

Definition. "Sample rate" is a frequency of "sampling" of a signal.

It is clear, that in engineering practice to restore a signal on the receiving side, it is more convenient to use not "functions – sampling", but "sampling" – instant values of a signal in points of "sampling" of a signal.

The class of functions with a limited spectrum is rather big. It is enough to note that spectra can be continuous and line.

Any periodic signal can be presented in the form:

$$s(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \dot{A}_n e^{jn\omega_1 t}$$

where

$$\dot{A}_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2}A_n e^{-j\varphi_n}$$

is the complex amplitudes of a spectrum containing the information both about amplitude and about phase spectra.

The spectrum of a periodic signal – a line spectrum is given:

$$\dot{S}_n = \frac{1}{2}\dot{A}_n T = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} s(t)e^{-\iota\omega t}dt$$

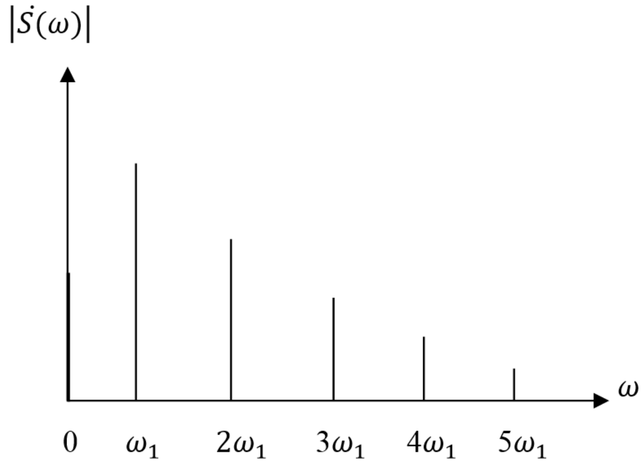

Fig. 1. The example of a line spectrum.

A **continuous** spectrum (a spectrum of a nonperiodic signal):

$$\dot{S}(\omega) = \int_{-\infty}^{\infty} s(t)e^{-\iota\omega t}dt$$

$S(\dot{\omega})$ is the complex spectrum that contains the information both on the spectrum of amplitudes and the spectrum of phases.

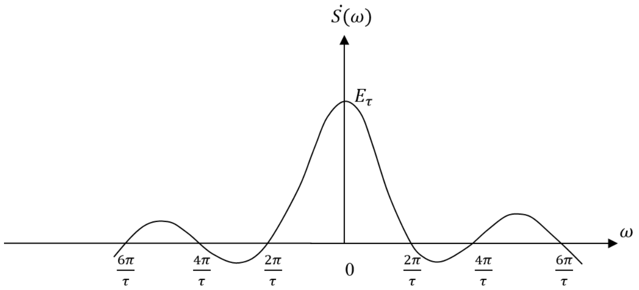

Fig. 2. The example of a continuous spectrum.

From the strict mathematical theory of a Fourier series [5], it is known that functions with a limited spectrum are representable on an infinite axis of time, and the function limited on time is representable on an infinite axis of frequencies (i.e., is not a function with a limited spectrum). In fact, from Figs. 1 and 2 given above it is clear that a periodic signal is the signal that is not limited on time has a spectrum in the form of a limited number of harmonicas. A nonperiodic signal is the signal limited on time has a spectrum, which is not limited on an axis of frequencies.

In engineering, data transmission (in television, in radio telemetry) when time of data transmission is obviously limited, theoretically we have to deal with the signals having an unlimited spectrum. At the same time, there are three engineering problems:

1. A problem of "assignment" of a cutoff frequency of a spectrum concerning which more high-frequency components can be neglected.

2. An engineering problem of approximate representation of the transmitted signal on the receiving side on its sampling (numbers) formed on the transmitting side.

3. A problem of assessment of an error of the representation.

V.A. Kotelnikov at the proof of the theorem applied an artificial mathematical method using a row [1], which was later called the Kotelnikov's row.

In engineering, at the representation of a signal on the receiving side the mathematical apparatus of Kotelnikov (as well as Shannon's "functions – sampling") that he offered at the proof of the theorem is not applied.

Thus, the sampling theorem for engineers is a mathematical justification of a possibility of representation (on a limited interval of time) of the signal consisting of an infinite number of points using a final quantity of numbers – sampling.

To represent a signal on the receiving side, a well-developed theory of approximation is used. In this theory, approximation methods using polynomials are employed. The Weierstrass theorem that he proved in the 19th century is the cornerstone of this method of approximation. One knows several methods of the proof of an approval of the Weierstrass theorem among which the most preferable for engineering is the proof provided in [7].

In relation to continuous functions of one valid variable set on a final piece *[a, b]*, the first Weierstrass theorem claims: for any continuous *f (x)* function on *[a, b]*, the sequence of ordinary polynomials which is evenly meeting on *[a, b]* to *f(x)* [7] exists.

The essence of the first Weierstrass theorem tells that any continuous function in the final closed interval can be spread out in evenly meeting row which members are polynomials.

The second Weierstrass theorem gives a clearer link of decomposition of a continuous periodic function with the 2π period (p. 40, [6]):

If *F(t)* is a continuous function with the 2π period, then whatever the number $\acute{\varepsilon} > 0$ is, there is the trigonometrical sum $S_n(t)$, $S_n(t) = a_0 + \Sigma (a_k \cos kt + b_k \sin kt)$, where summing is carried out by k measured from 1 to *n*:

$$[n = n(\acute{\varepsilon})].$$

This sum for all t meets the inequality

$$|F(t) - Sn(t)| \le \acute{\varepsilon}.$$

As it is known, polynomials relate to the class of integer functions [8].

Polynomials, an exponential function, trigonometrical functions (sine and cosine), and others belong to the integer functions. The integer functions, which are not polynomials (they are called transcendental), in many respects behave as some kind of "polynomials of an infinitely high degree". In engineering, the approximation of functions (signals) with use of polynomials of a final degree is of interest.

From the representatives of the integer functions we will concern only research of polynomials of a final degree and trigonometrical functions.

In the introduction to Ya.I. Khurgin and V.P. Yakovlev [8] book, it is written that the sampling theorem "is the theorem of an opportunity for signal transferring with a limited spectrum in principle to use not all its values but only separate periodically chosen ("equidistant", *author's note*) values, and at the same time to restore unambiguously a signal on the receiving side on the entire time axis".

In addition, the theorem of special not equidistant "sampling", or the theorem of sampling in special points proving a possibility to use "special" sampling for signal restoration is known.

Special readings are the values of a signal taken in special points of a signal where the first, second, and other derivatives are equal to zero [10, 11], and equidistant readings are considered as a special case of not equidistant readings. At the same time to restore a signal by not equidistant readings, polynomials of a final degree are also used.

Further in V.P. Khurgin's preface to [9] we read: "However soon it was succeeded to understand: all the matter is that functions with a limited range are the whole analytical functions, and, therefore, Kotelnikov's formula is one of the possible interpolation formulas used in the theory of the integer functions". Moreover, below it is about the proof and generalization by students B.S. Tsybakov and V.P. Yakovlev of the sampling theorem "by means of the instrument of the theory of interpolating of the integer functions".

However, the analysis of literature showed that B.S. Tsybakov and V.P. Yakovleva's publications with the proof of this statement are absent, or proofs are published in the little-known edition by a small circulation.

To restore the specified gap, it is of interest to prove or disprove five statements and to specify living conditions of these statements:

1. Signals, mathematically representable on a final piece *[a, b]* in the form of polynomials of a final degree are functions with a limited spectrum.

2. Signals, mathematically representable on a final piece *[a, b]* in the form of trigonometrical functions (a sine and a cosine) are functions with a limited spectrum.

3. Signals, mathematically representable on a final piece *[a, b]* in the form of a piece of an exponential function, are functions with a limited spectrum.

4. Signals, mathematically representable on a final piece *[a, b]* in the form of integer functions are functions with a limited spectrum. In view of complexity, it is possible to leave this fourth (general-theoretical) statement without proof in view of taking into account the practical importance for engineering practice of the first three statements (theorems).

5. The continuous signal presented on the transmitting side in the form of a polynomial of a final degree is a signal with a limited spectrum, and W depends on coefficients of a polynomial and is defined by the following expression:

$$s(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \dot{A}_n e^{jn\omega_1 t}$$

At the practical engineering approach the proof of three first private statements, living conditions of these statements, and founding is of interest to these three cases of the cutoff frequency of W and "numbers" that is analogs to numbers "sampling" in the sampling theorem. We will note that K. Shannon offered the term "sampling", and V.A. Kotelnikov used the term "numbers".

It is clear, that the statement is justified: if on a final piece *[a, b]* a mathematical representation of a signal in the form of the integer function (polynomial, trigonometrical, and indicative) is known, thus at a priori knownness of a general view of the integer function on the transmitting side, a signal can be transferred by transfer of a final quantity of the numbers representing the coefficients of this integer function.

The mathematical apparatus of decomposition of functions in orthogonal bases is known [12]. A mathematical language of pure mathematicians [12] demands essential efforts for adaptation (translation) of this language into the language clear to the practical engineers who are engaged in development of algorithms of preliminary data processing on the transmitting side and the hardware of communication realizing these algorithms.

If the orthogonal basis has a final quantity of orthogonal axes of basis of decomposition, then representation of a signal with a limited range supports the final number of members of decomposition in this basis.

A final, without the rest, number of members of representation of the function continuous on a final piece *[a, b]*, also represents the main idea of a possibility of representation of this function with the use of a final quantity of numbers.

It is possible to use a concept of the "generalized limited range" outlined in [10].

The polynomial of a final degree represents the decomposition of a signal on the degrees representing decomposition in the orthogonal basis *(1, x, x2, x3 …)*.

If the coefficients of decomposition members are arranged on the extent of decrease on the absolute value, then the size of coefficient is an analog to the amplitude, and the smallest decomposition coefficient is a certain analog to "the boundary frequency" of *W* for the signals with the generalized limited range.

Conclusion

1. The statement that "functions with a limited range are integer analytical functions" [8], contains either inaccuracy or a slip. Actually, one should speak about a narrower class of the integer analytical functions – about analytical functions with the final number of members. And if it is about polynomials, then about polynomials of final degree especially as the formula (row) of Kotelnikov supports the final number of members (this private or general statement should be proved or disproved in the theorem 1).

2. A continuous function with a limited range on *[a, b]* using sampling can be presented (is approximated) with apply of a polynomial of a final degree.

3. Representation (approximation) on *[a, b]* of continuous signals employing a polynomial of a final degree sets tasks to evaluate an error of approximation and the choice–justification (depending on a preset value of an error of approximation) of the maximum degree of the approximating polynomial.

References

1. Kotel'nikov V.A. O propusknoy sposobnosti "efira" i provoloki v elektrosvyazi [On the capacity of ether and wire in telecommunication]. Radiotekhnika [Radioengineering], 1995, No. 4–5 (double), pp. 42–55. (in Russian)
2. Shannon C. Raboty po teorii informatsii i kibernetike [Works on Information and Cybernetics]. Moscow, Izdatel'stvo inostrannoy literatury, 1963, pp. 243–332. (translated from English)
3. Jerry J.A. Teorema otschetov Shennona, ee razlichnye obobshcheniya i prilozheniya [The Shannon Sampling Theorem — Its Various Extension And Application]. TIIER, 1977, Vol. 65, No. 11, pp. 53–89. (in Russian)
4. Goldman S. Teoriya informatsii [Information theory]. Moscow, Izdatel'stvo inostrannoy literatury, 1957, 446 p. (translated from English)
5. Akhiezer N.I. Lektsii po teorii approksimatsii [Lectures on approximation theory]. Moscow, OGIZ, GIT-TL, 1947, 324 p. (in Russian)
6. Gnedenko B.V. Vvedenie v spetsial'nost' "matematika" [Introduction to mathematics]. Moscow, Nauka, 1991. (in Russian)
7. Markushevich A.I. Tselye funktsii. 2-oe izdanie [Entire functions]. 2nd edition. Moscow, Nauka, 1975, 119 p. (in Russian)
8. Khurgin Ya.I., Yakovlev V.P. Metody teorii tselykh funktsiy v radiotekhnike, teorii svyazi i optike [Methods of the theory of entire functions in radio engineering, communication theory, and optics]. Moscow, Fizmatgiz, 1962, 220 p. (in Russian)
9. Pobedonostsev V.A. Osnovaniya informmetrii [Informetrics foundations]. Moscow, Radio i svyaz', 2000, 193 p. (in Russian)
10. Pobedonostsev V.A. Teorema o neravnootstoyashchikh otschetakh [The theorem on unequally spaced readings]. Radiotekhnicheskie tetradi. Spets. vyp. [Radio electronics booklets Special edition]. Moscow, MEI, 1995, No. 8, pp. 25–27. (in Russian)
11. Pobedonostsev V.A. Teoreticheskie voprosy izmereniya kolichestva informatsii nepreryvnykh signalov na konechnykh intervalakh [Theoretical Questions in Measuring the Amount of Information in Continuous Signals on Finite Intervals]. Raketno-kosmicheskoe priborostroenie i informatsionnye sistemy [Rocket-Space Device Engineering and Information Systems], 2014, Vol. 1, No. 2, pp. 47–58. (in Russian)
12. Razmakhnin M.K, Yakovlev V.P. Funktsii s dvoynoy ortogonal'nost'yu v radioelektronike i optike [Functions in double ortogonality in radio electronics and optics]. Moscow, "Sov. Radio", 1971, 256 p. (in Russian)